Y2P9 XMQs and MS

(Total: 162 marks)

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 1. P1_Sample    Q10.  5 marks - Y2P9 Differentiation

 2. P1_Sample    Q13. 13 marks - Y2P8 Parametric equations

 3. P1_Sample    Q15.  8 marks - Y2P9 Differentiation

 4. P2_Sample    Q3 .  4 marks - Y2P9 Differentiation

 5. P2_Sample    Q5 .  4 marks - Y1P14 Exponentials and logarithms

 6. P2_Specimen  Q12.  7 marks - Y2P9 Differentiation

 7. P1_2018      Q5 .  5 marks - Y2P9 Differentiation

 8. P1_2018      Q9 . 10 marks - Y2P9 Differentiation

 9. P2_2018      Q9 .  5 marks - Y2P9 Differentiation

10. P2_2018      Q14. 10 marks - Y2P9 Differentiation

11. P1_2019      Q3 .  5 marks - Y2P9 Differentiation

12. P1_2019      Q12. 10 marks - Y2P9 Differentiation

13. P1_2019      Q14.  7 marks - Y2P9 Differentiation

14. P1_2020      Q9 .  9 marks - Y2P9 Differentiation

15. P1_2020      Q15.  7 marks - Y2P9 Differentiation

16. P2_2020      Q13.  6 marks - Y2P9 Differentiation

17. P1_2021      Q14.  4 marks - Y2P9 Differentiation

18. P2_2021      Q8 .  9 marks - Y2P9 Differentiation

19. P2_2021      Q13.  6 marks - Y2P8 Parametric equations

20. P1_2022      Q15. 10 marks - Y2P9 Differentiation

21. P2_2022      Q12.  6 marks - Y2P9 Differentiation

22. P2_2022      Q16. 12 marks - Y2P9 Differentiation
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10. Given that θ is measured in radians, prove, from first principles, that the derivative of $\sin\theta$ is $\cos\theta$

You may assume the formula for $\sin(A \pm B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

(Total for Question 10 is 5 marks)

Question	Scheme	Marks	AOs
10	Use of $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}$	B1	2.1
	Uses the compound angle identity for $\sin(A+B)$ with $A=\theta,\ B=h$ $\Rightarrow \sin(\theta+h)=\sin\theta\cos h+\cos\theta\sin h$	M1	1.1b
	Achieves $\dfrac{\sin(\theta+h)-\sin\theta}{h}=\dfrac{\sin\theta\cos h+\cos\theta\sin h-\sin\theta}{h}$	A1	1.1b
	$=\dfrac{\sin h}{h}\cos\theta+\left(\dfrac{\cos h-1}{h}\right)\sin\theta$	M1	2.1
	Uses $h\to 0,\ \dfrac{\sin h}{h}\to 1$ and $\dfrac{\cos h-1}{h}\to 0$ Hence the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$ *	A1*	2.5
		(5 marks)	

Notes:

B1: States or implies that the gradient of the chord is $\dfrac{\sin(\theta+h)-\sin\theta}{h}$ or similar such as $\dfrac{\sin(\theta+\delta\theta)-\sin\theta}{\theta+\delta\theta-\theta}$ for a small h or $\delta\theta$

M1: Uses the compound angle identity for $\sin(A+B)$ with $A=\theta,\ B=h$ or $\delta\theta$

A1: Obtains $\dfrac{\sin\theta\cos h+\cos\theta\sin h-\sin\theta}{h}$ or equivalent

M1: Writes their expression in terms of $\dfrac{\sin h}{h}$ and $\dfrac{\cos h-1}{h}$

A1*: Uses correct language to explain that $\dfrac{dy}{d\theta}=\cos\theta$

For this method they should use all of the given statements $h\to 0,\ \dfrac{\sin h}{h}\to 1$, $\dfrac{\cos h-1}{h}\to 0$ meaning that the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$

and therefore the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$

Question	Scheme	Marks	AOs
10alt	Use of $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}$	B1	2.1
	Sets $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\dfrac{\sin\left(\theta+\frac{h}{2}+\frac{h}{2}\right)-\sin\left(\theta+\frac{h}{2}-\frac{h}{2}\right)}{h}$ and uses the compound angle identity for sin(*A* + *B*) and sin(*A* – *B*) with $A=\theta+\frac{h}{2},\quad B=\frac{h}{2}$	M1	1.1b
	Achieves $\dfrac{\sin(\theta+h)-\sin\theta}{h}=$ $\dfrac{\left[\sin\left(\theta+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)+\cos\left(\theta+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]-\left[\sin\left(\theta+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)-\cos\left(\theta+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]}{h}$	A1	1.1b
	$=\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\cos\left(\theta+\frac{h}{2}\right)$	M1	2.1
	Uses $h\to 0,\ \frac{h}{2}\to 0$ hence $\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\to 1$ and $\cos\left(\theta+\frac{h}{2}\right)\to\cos\theta$ Therefore the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and the gradient of the chord $\to$ gradient of the curve $\Rightarrow\dfrac{dy}{d\theta}=\cos\theta$ *	A1*	2.5
		(5 marks)	

Additional notes:

A1*: Uses correct language to explain that $\dfrac{dy}{d\theta}=\cos\theta$. For this method they should use the (adapted) given statement $h\to 0,\ \frac{h}{2}\to 0$ hence $\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\to 1$ with $\cos\left(\theta+\frac{h}{2}\right)\to\cos\theta$ meaning that the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and therefore the gradient of the chord $\to$ gradient of the curve $\Rightarrow\dfrac{dy}{d\theta}=\cos\theta$

13. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of t.

(2)

The point P lies on C where $t = \dfrac{2\pi}{3}$

The line l is the normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.

You must show clearly how you obtained your answers.

(6)

Question	Scheme	Marks	AOs
13(a)	Attempts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$	M1	1.1b
	$\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} \quad (= 2\sqrt{3}\cos t)$	A1	1.1b
		(2)	
(b)	Substitutes $t = \frac{2\pi}{3}$ in $\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} = (-\sqrt{3})$	M1	2.1
	Uses gradient of normal = $-\frac{1}{dy/dx} = \left(\frac{1}{\sqrt{3}}\right)$	M1	2.1
	Coordinates of $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$	B1	1.1b
	Correct form of normal $y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1)$	M1	2.1
	Completes proof $\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ *	A1*	1.1b
		(5)	
(c)	Substitutes $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$	M1	3.1a
	Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic in $\cos t$	M1	3.1a
	$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$	A1	1.1b
	Finds $\cos t = \frac{5}{6}, \cancel{-\frac{1}{2}}$	M1	2.4
	Substitutes their $\cos t = \frac{5}{6}$ into $x = 2\cos t,\ y = \sqrt{3}\cos 2t$,	M1	1.1b
	$Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$	A1	1.1b
		(6)	
			(13 marks)

Question 13 continued

Notes:

(a)

M1: Attempts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and achieves a form $k\frac{\sin 2t}{\sin t}$ Alternatively candidates may apply the double angle identity for $\cos 2t$ and achieve a form $k\frac{\sin t \cos t}{\sin t}$

A1: Scored for a correct answer, either $\frac{\sqrt{3}\sin 2t}{\sin t}$ or $2\sqrt{3}\cos t$

(b)

M1: For substituting $t = \frac{2\pi}{3}$ in their $\frac{dy}{dx}$ which must be in terms of t

M1: Uses the gradient of the normal is the negative reciprocal of the value of $\frac{dy}{dx}$. This may be seen in the equation of l.

B1: States or uses (in their tangent or normal) that $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$

M1: Uses their numerical value of $-1 \Big/ \frac{dy}{dx}$ with their $\left(-1, -\frac{\sqrt{3}}{2}\right)$ to form an equation of the normal at P

A1*: This is a proof and all aspects need to be correct. Correct answer only $2x - 2\sqrt{3}y - 1 = 0$

(c)

M1: For substituting $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$ to produce an equation in t. Alternatively candidates could use $\cos 2t = 2\cos^2 t - 1$ to set up an equation of the form $y = Ax^2 + B$.

M1: Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic equation in $\cos t$
In the alternative method it is for combining their $y = Ax^2 + B$ with $2x - 2\sqrt{3}y - 1 = 0$ to get an equation in just one variable

A1: For the correct quadratic equation $12\cos^2 t - 4\cos t - 5 = 0$
Alternatively the equations in x and y are $3x^2 - 2x - 5 = 0$ $\quad 12\sqrt{3}y^2 + 4y - 7\sqrt{3} = 0$

M1: Solves the quadratic equation in $\cos t$ (or x or y) and rejects the value corresponding to P.

M1: Substitutes their $\cos t = \frac{5}{6}$ or their $t = \arccos\left(\frac{5}{6}\right)$ in $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$
If a value of x or y has been found it is for finding the other coordinate.

A1: $Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$. Allow $x = \frac{5}{3}, y = \frac{7}{18}\sqrt{3}$ but do not allow decimal equivalents.

15.

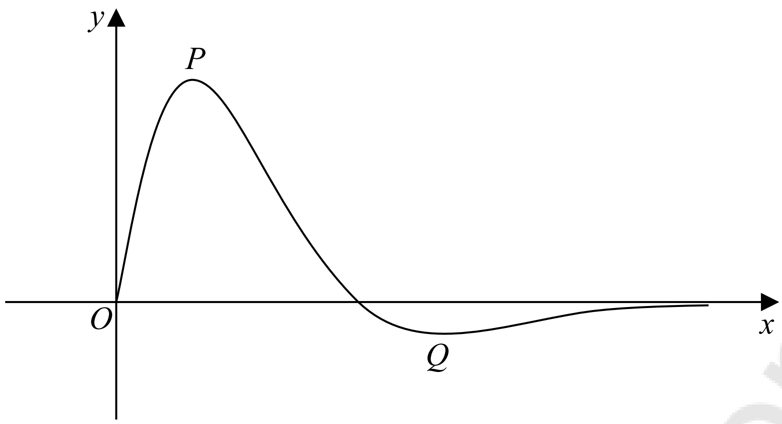


Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{4\sin 2x}{\mathrm{e}^{\sqrt{2}x-1}}, \quad 0 \leqslant x \leqslant \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q as shown in Figure 5.

(a) Show that the x coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2}$$

(4)

(b) Using your answer to part (a), find the x-coordinate of the minimum turning point on the curve with equation

(i) $y = \mathrm{f}(2x)$.

(ii) $y = 3 - 2\mathrm{f}(x)$.

(4)

Question	Scheme	Marks	AOs
15(a)	Attempts to differentiate using the quotient rule or otherwise	M1	2.1
	$f'(x)=\dfrac{e^{\sqrt{2}x-1}\times 8\cos 2x-4\sin 2x\times\sqrt{2}e^{\sqrt{2}x-1}}{\left(e^{\sqrt{2}x-1}\right)^2}$	A1	1.1b
	Sets $f'(x)=0$ and divides/ factorises out the $e^{\sqrt{2}x-1}$ terms	M1	2.1
	Proceeds via $\dfrac{\sin 2x}{\cos 2x}=\dfrac{8}{4\sqrt{2}}$ to $\Rightarrow \tan 2x=\sqrt{2}$ *	A1*	1.1b
		(4)	
(b)	(i) Solves $\tan 4x=\sqrt{2}$ and attempts to find the 2nd solution	M1	3.1a
	$x=1.02$	A1	1.1b
	(ii) Solves $\tan 2x=\sqrt{2}$ and attempts to find the 1st solution	M1	3.1a
	$x=0.478$	A1	1.1b
		(4)	
		(8 marks)	

Notes:

(a)

M1: Attempts to differentiate by using the quotient rule with $u=4\sin 2x$ and $v=e^{\sqrt{2}x-1}$ or alternatively uses the product rule with $u=4\sin 2x$ and $v=e^{1-\sqrt{2}x}$

A1: For achieving a correct $f'(x)$. For the product rule

$f'(x)=e^{1-\sqrt{2}x}\times 8\cos 2x+4\sin 2x\times -\sqrt{2}e^{1-\sqrt{2}x}$

M1: This is scored for cancelling/ factorising out the exponential term. Look for an equation in just $\cos 2x$ and $\sin 2x$

A1*: Proceeds to $\tan 2x=\sqrt{2}$. This is a given answer.

(b) (i)

M1: Solves $\tan 4x=\sqrt{2}$ attempts to find the 2nd solution. Look for $x=\dfrac{\pi+\arctan\sqrt{2}}{4}$

Alternatively finds the 2nd solution of $\tan 2x=\sqrt{2}$ and attempts to divide by 2

A1: Allow awrt $x=1.02$. The correct answer, with no incorrect working scores both marks

(b)(ii)

M1: Solves $\tan 2x=\sqrt{2}$ attempts to find the 1st solution. Look for $x=\dfrac{\arctan\sqrt{2}}{2}$

A1: Allow awrt $x=0.478$. The correct answer, with no incorrect working scores both marks

3. Given $y = x(2x + 1)^4$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x+1)^n(Ax+B)$$

where n, A and B are constants to be found.

(4)

(Total for Question 3 is 4 marks)

Question	Scheme	Marks	AOs
3	Attempts the product and chain rule on $y = x(2x+1)^4$	M1	2.1
	$\frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$	A1	1.1b
	Takes out a common factor $\frac{dy}{dx} = (2x+1)^3 \{(2x+1)+8x\}$	M1	1.1b
	$\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n=3, A=10, B=1$	A1	1.1b
			(4 marks)

Notes:

M1: Applies the product rule to reach $\frac{dy}{dx} = (2x+1)^4 + Bx(2x+1)^3$

A1: $\frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$

M1: Takes out a common factor of $(2x+1)^3$

A1: The form of this answer is given. Look for $\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n=3, A=10, B=1$

5. The mass, m grams, of a radioactive substance, t years after first being observed, is modelled by the equation

$$m = 25e^{-0.05t}$$

According to the model,

(a) find the mass of the radioactive substance six months after it was first observed,

(2)

(b) show that $\frac{dm}{dt} = km$, where k is a constant to be found.

(2)

(Total for Question 5 is 4 marks)

Question	Scheme	Marks	AOs
5(a)	Substitutes $t = 0.5$ into $m = 25e^{-0.05t} \Rightarrow m = 25e^{-0.05\times 0.5}$	M1	3.4
	$\Rightarrow m = 24.4\text{g}$	A1	1.1b
		(2)	
(b)	States or uses $\frac{d}{dt}\left(e^{-0.05t}\right) = \pm C e^{-0.05t}$	M1	2.1
	$\frac{dm}{dt} = -0.05 \times 25e^{-0.05t} = -0.05m \Rightarrow k = -0.05$	A1	1.1b
		(2)	
			(4 marks)

Notes:

(a)

M1: Substitutes $t = 0.5$ into $m = 25e^{-0.05t} \Rightarrow m = 25e^{-0.05\times 0.5}$

A1: $m = 24.4\text{g}$ An answer of $m = 24.4\text{g}$ with no working would score both marks

(b)

M1: Applies the rule $\frac{d}{dt}\left(e^{kx}\right) = k\,e^{kx}$ in this context by stating or using $\frac{d}{dt}\left(e^{-0.05t}\right) = \pm C e^{-0.05t}$

A1: $\frac{dm}{dt} = -0.05 \times 25e^{-0.05t} = -0.05m \Rightarrow k = -0.05$

12. A curve C is given by the equation

$$\sin x + \cos y = 0.5 \qquad -\frac{\pi}{2} \leqslant x < \frac{3\pi}{2}, -\pi < y < \pi$$

A point P lies on C.

The tangent to C at the point P is parallel to the x-axis.

Find the exact coordinates of all possible points P, justifying your answer.
(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(7)

Question	Scheme	Marks	AOs
12	Complete process to find at least one set of coordinates for *P*. The process must include evidence of • differentiating • setting $\frac{dy}{dx} = 0$ to find $x = ...$ • substituting $x = ...$ into $\sin x + \cos y = 0.5$ to find $y = ...$	M1	3.1a
	$\left\{\cancel{\frac{dy}{dx}\times}\right\}$ $\cos x - \sin y\frac{dy}{dx} = 0$	B1	1.1b
	Applies $\frac{dy}{dx} = 0$ (e.g. $\cos x = 0$ or $\frac{\cos x}{\sin y} = 0 \Rightarrow \cos x = 0) \Rightarrow x = ...$	M1	2.2a
	giving at least one of either $x = -\frac{\pi}{2}$ or $x = \frac{\pi}{2}$	A1	1.1b
	$x = \frac{\pi}{2} \Rightarrow \sin\left(\frac{\pi}{2}\right) + \cos y = 0.5 \Rightarrow \cos y = -\frac{1}{2} \Rightarrow y = \frac{2\pi}{3}$ or $-\frac{2\pi}{3}$	M1	1.1b
	So in specified range, $(x, y) = \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$, by cso	A1	1.1b
	$x = -\frac{\pi}{2} \Rightarrow \sin\left(-\frac{\pi}{2}\right) + \cos y = 0.5 \Rightarrow \cos y = 1.5$ has no solutions, and so there are exactly 2 possible points *P*.	B1	2.1
		(7)	
			(7 marks)

Question 12 Notes:

M1:	See scheme
B1:	Correct differentiated equation. E.g. $\cos x - \sin y\frac{dy}{dx} = 0$
M1:	Uses the information "the tangent to *C* at the point *P* is parallel to the *x*-axis" to deduce and apply $\frac{dy}{dx} = 0$ and finds $x = ...$
A1:	See scheme
M1:	For substituting one of their values from $\frac{dy}{dx} = 0$ into $\sin x + \cos y = 0.5$ and so finds $x = ..., y = ...$
A1:	Selects coordinates for *P* on *C* satisfying $\frac{dy}{dx} = 0$ and $-\frac{\pi}{2} \leqslant x < \frac{3\pi}{2}$, $-\pi < y < \pi$ i.e. finds $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$ and no other points by correct solution only
B1:	Complete argument to show that there are exactly 2 possible points *P*.

5. Given that

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1 + \sin 2\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where A is a rational constant to be found.

(5)

Question	Scheme	Marks	AOs
5	$\dfrac{dy}{d\theta}=\dfrac{(2\sin\theta+2\cos\theta)3\cos\theta-3\sin\theta(2\cos\theta-2\sin\theta)}{(2\sin\theta+2\cos\theta)^2}$	M1 A1	1.1b 1.1b
	Expands and uses $\sin^2\theta+\cos^2\theta=1$ at least once in the numerator or the denominator or uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow \dfrac{dy}{d\theta}=\dfrac{...}{.........C\sin\theta\cos\theta}$	M1	3.1a
	Expands and uses $\sin^2\theta+\cos^2\theta=1$ the numerator and the denominator AND uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow \dfrac{dy}{d\theta}=\dfrac{P}{Q+R\sin 2\theta}$	M1	2.1
	$\Rightarrow \dfrac{dy}{d\theta}=\dfrac{3}{2+2\sin 2\theta}=\dfrac{3/2}{1+\sin 2\theta}$	A1	1.1b
	(5 marks)		

Notes:

M1: For choosing either the quotient, product rule or implicit differentiation and applying it to the given function. Look for the correct form of $\dfrac{dy}{d\theta}$ (condone it being stated as $\dfrac{dy}{dx}$) but tolerate slips on the coefficients and also condone $\dfrac{d(\sin\theta)}{d\theta}=\pm\cos\theta$ and $\dfrac{d(\cos\theta)}{d\theta}=\pm\sin\theta$

For quotient rule look for $\dfrac{dy}{d\theta}=\dfrac{(2\sin\theta+2\cos\theta)\times\pm...\cos\theta-3\sin\theta(\pm...\cos\theta\pm...\sin\theta)}{(2\sin\theta+2\cos\theta)^2}$

For product rule look for

$$\frac{dy}{d\theta}=(2\sin\theta+2\cos\theta)^{-1}\times\pm...\cos\theta\pm3\sin\theta\times(2\sin\theta+2\cos\theta)^{-2}\times(\pm...\cos\theta\pm...\sin\theta)$$

Implicit differentiation look for $(...\cos\theta\pm...\sin\theta)y+(2\sin\theta+2\cos\theta)\dfrac{dy}{d\theta}=...\cos\theta$

A1: A correct expression involving $\dfrac{dy}{d\theta}$ condoning it appearing as $\dfrac{dy}{dx}$

M1: Expands and uses $\sin^2\theta+\cos^2\theta=1$ at least once in the numerator or the denominator OR uses $2\sin\theta\cos\theta=\sin 2\theta$ in $\Rightarrow \dfrac{dy}{d\theta}=\dfrac{...}{.........C\sin\theta\cos\theta}$

M1: Expands and uses $\sin^2\theta+\cos^2\theta=1$ in the numerator and the denominator AND uses $2\sin\theta\cos\theta=\sin 2\theta$ in the denominator to reach an expression of the form $\dfrac{dy}{d\theta}=\dfrac{P}{Q+R\sin 2\theta}$.

A1: Fully correct proof with $A=\dfrac{3}{2}$ stated but allow for example $\dfrac{3/2}{1+\sin 2\theta}$

Allow recovery from missing brackets. Condone notation slips. This is not a given answer

9.

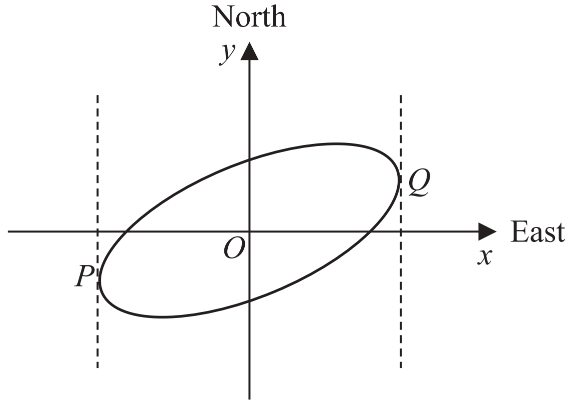


Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

(a) Show that $\frac{dy}{dx} = \frac{y - x}{3y - x}$

(4)

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P.

(5)

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O. (You **do not** need to carry out this calculation).

(1)

Question	Scheme	Marks	AOs
9(a)	Either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$	M1	2.1
	$2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$	A1	1.1b
	$(6y-2x)\frac{dy}{dx}=2y-2x$	M1	2.1
	$\frac{dy}{dx}=\frac{2y-2x}{6y-2x}=\frac{y-x}{3y-x}$ *	A1*	1.1b
		(4)	
(b)	$\left(\text{At } P \text{ and } Q\ \frac{dy}{dx}\to\infty\Rightarrow\right)$ Deduces that $3y-x=0$	M1	2.2a
	Solves $y=\frac{1}{3}x$ and $x^2-2xy+3y^2=50$ simultaneously	M1	3.1a
	$\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$	A1	1.1b
	Using $y=\frac{1}{3}x \Rightarrow x=..$ AND $y=..$	dM1	1.1b
	$P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$	A1	2.2a
		(5)	
(c)	Explains that you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution	B1ft	2.4
		(1)	
			(10 marks)

Notes:
(a)

M1: For selecting the appropriate method of differentiating either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$

It may be quite difficult awarding it for the product rule but condone $-2xy \to -2x\frac{dy}{dx}+2y$ unless you see evidence that they have used the incorrect law *vu*'-*uv*'

A1: Fully correct derivative $2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

Allow attempts where candidates write $2x\,dx-2x\,dy-2y\,dx+6y\,dy=0$

but watch for students who write $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}$ This, on its own, is A0 unless you are convinced that this is just their notation. Eg $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

M1: For a valid attempt at making $\frac{dy}{dx}$ the subject. with two terms in $\frac{dy}{dx}$ coming from $3y^2$ and $2xy$

Look for $(...\pm...)\frac{dy}{dx}=......$ It is implied by $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$

This cannot be scored from attempts such as $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y$ which only has one correct term.

A1*: $\frac{dy}{dx}=\frac{y-x}{3y-x}$ with no errors or omissions.

The previous line $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$ or equivalent must be seen.

(b)

M1: Deduces that $3y-x=0$ oe

M1: Attempts to find either the *x* or *y* coordinates of *P* and *Q* by solving their $y=\frac{1}{3}x$ with $x^2-2xy+3y^2=50$ simultaneously. Allow for finding a quadratic equation in *x* or *y* and solving to find at least one value for *x* or *y*.

This may be awarded when candidates make the numerator = 0 ie using $y=x$

A1: $\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$

dM1: Dependent upon the previous M, it is for finding the *y* coordinate from their *x* (or vice versa)
This may also be scored following the numerator being set to 0 ie using $y=x$

A1: Deduces that $P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$ OE. Allow to be *x* =... *y* =...

(c)

B1ft: Explains that this is where $\frac{dy}{dx}=0$ and so you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution (or larger solution).

Allow a follow through for candidates who mix up parts (b) and (c)

Alternatively candidates could complete the square $(x-y)^2+2y^2=50$ and state that *y* would reach a maximum value when $x=y$ and choose the positive solution from $2y^2=50$

9. Given that θ is measured in radians, prove, from first principles, that

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for $\cos(A \pm B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

Question	Scheme	Marks	AOs
9	$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$; as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$		
	$\frac{\cos(\theta+h)-\cos\theta}{h}$	B1	2.1
	$= \frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$	M1	1.1b
		A1	1.1b
	$= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$		
	As $h \to 0$, $-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta \to -1\sin\theta + 0\cos\theta$	dM1	2.1
	so $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$ *	A1*	2.5
		(5)	
			(5 marks)

Notes for Question 9

B1:	Gives the correct fraction such as $\frac{\cos(\theta+h)-\cos\theta}{h}$ or $\frac{\cos(\theta+\delta\theta)-\cos\theta}{\delta\theta}$ Allow $\frac{\cos(\theta+h)-\cos\theta}{(\theta+h)-\theta}$ o.e. **Note:** $\cos(\theta+h)$ or $\cos(\theta+\delta\theta)$ may be expanded
M1:	Uses the compound angle formula for $\cos(\theta+h)$ to give $\cos\theta\cos h \pm \sin\theta\sin h$
A1:	Achieves $\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$ or equivalent
dM1:	**dependent on both the B and M marks being awarded** Complete attempt to apply the given limits to the gradient of their chord
Note:	They must isolate $\frac{\sin h}{h}$ and $\left(\frac{\cos h - 1}{h}\right)$, and replace $\frac{\sin h}{h}$ with 1 and replace $\left(\frac{\cos h - 1}{h}\right)$ with 0
A1*:	cso. Uses correct mathematical language of limiting arguments to prove $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$
Note:	Acceptable responses for the final A mark include: • $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta\right) = -1\sin\theta + 0\cos\theta = -\sin\theta$ • Gradient of chord $= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$. As $h \to 0$, gradient of chord tends to the gradient of the curve, so derivative is $-\sin\theta$ • Gradient of chord $= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$. As $h \to 0$, gradient of ***curve*** is $-\sin\theta$
Note:	Give final A0 for the following example which shows ***no limiting arguments***: when $h = 0$, $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta = -1\sin\theta + 0\cos\theta = -\sin\theta$
Note:	Do not allow the final A1 for stating $\frac{\sin h}{h} = 1$ or $\left(\frac{\cos h - 1}{h}\right) = 0$ and attempting to apply these
Note:	In this question $\delta\theta$ may be used in place of h
Note:	Condone $\mathrm{f}'(\theta)$ where $\mathrm{f}(\theta) = \cos\theta$ or $\frac{\mathrm{dy}}{\mathrm{d}\theta}$ where $y = \cos\theta$ used in place of $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)$

Notes for Question 9 Continued	
Note:	Condone x used in place of θ if this is done consistently
Note:	Give final A0 for • $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos x) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta\right) = -1\sin\theta + 0\cos\theta = -\sin\theta$ • $\frac{\mathrm{d}}{\mathrm{d}\theta} = \ldots$ • Defining $\mathrm{f}(x) = \cos\theta$ and applying $\mathrm{f}'(x) = \ldots$ • $\frac{\mathrm{d}}{\mathrm{d}x}(\cos\theta)$
Note:	Give final A1 for a correct limiting argument in x, followed by $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$ e.g. $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos x) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin x + \left(\frac{\cos h - 1}{h}\right)\cos x\right) = -1\sin x + 0\cos x = -\sin x$ $\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$
Note:	Applying $h \to 0,\ \sin h \to h,\ \cos h \to 1$ to give e.g. $\lim_{h\to 0}\left(\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}\right) = \left(\frac{\cos\theta(1) - \sin\theta(h) - \cos\theta}{h}\right) = \frac{-\sin\theta(h)}{h} = -\sin\theta$ is final M0 A0 for incorrect application of limits
Note:	$\lim_{h\to 0}\left(\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}\right) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta\right)$ $= \underleftrightarrow{\lim_{h\to 0}\left(-(1)\sin\theta + 0\cos\theta\right)} = -\sin\theta.$ So for $\underleftrightarrow{\text{not removing } \lim_{h\to 0}}$ when the limit was taken is final A0
Note:	**<u>Alternative Method:</u>** Considers $\frac{\cos(\theta + h) - \cos(\theta - h)}{(\theta + h) - (\theta - h)}$ which simplifies to $\frac{-2\sin\theta\sin h}{2h}$

14. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1)

Question	Scheme	Marks	AOs
14	$N=\dfrac{900}{3+7e^{-0.25t}}=900(3+7e^{-0.25t})^{-1},\ t\in\mathbb{R},\ t\geq 0\ ;\quad \dfrac{dN}{dt}=\dfrac{N(300-N)}{1200}$		
(a)	90	B1	3.4
		(1)	
(b) Way 1	$\dfrac{dN}{dt}=-900(3+7e^{-0.25t})^{-2}\left(7(-0.25)e^{-0.25t}\right)\ \left\{=\dfrac{900(0.25)(7)e^{-0.25t}}{(3+7e^{-0.25t})^2}\right\}$	M1	2.1
		A1	1.1b
	$\Rightarrow \dfrac{dN}{dt}=\dfrac{900(0.25)\left(\left(\dfrac{900}{N}-3\right)\right)}{\left(\dfrac{900}{N}\right)^2}$	dM1	2.1
	correct algebra leading to $\dfrac{dN}{dt}=\dfrac{N(300-N)}{1200}$ *	A1*	1.1b
		(4)	
(b) Way 2	$\dfrac{dN}{dt}=-900(3+7e^{-0.25t})^{-2}\left(7(-0.25)e^{-0.25t}\right)\ \left\{=\dfrac{900(0.25)(7)e^{-0.25t}}{(3+7e^{-0.25t})^2}\right\}$	M1	2.1
		A1	1.1b
	$\dfrac{N(300-N)}{1200}=\dfrac{\left(\dfrac{900}{3+7e^{-0.25t}}\right)\left(300-\dfrac{900}{3+7e^{-0.25t}}\right)}{1200}$	dM1	2.1
	LHS $=\dfrac{1575e^{-0..25t}}{(3+7e^{-0.25t})^2}$ o.e., RHS $=\dfrac{900(300(3+7e^{-0.25t})-900)}{1200(3+7e^{-0.25t})^2}=\dfrac{1575e^{-0..25t}}{(3+7e^{-0.25t})^2}$ o.e. and states hence $\dfrac{dN}{dt}=\dfrac{N(300-N)}{1200}$ (or LHS = RHS) *	A1*	1.1b
		(4)	
(c)	Deduces $N=150$ (can be implied)	B1	2.2a
	so $150=\dfrac{900}{3+7e^{-0.25T}}\ \Rightarrow\ e^{-0.25T}=\dfrac{3}{7}$	M1	3.4
	$T=-4\ln\left(\dfrac{3}{7}\right)$ or $T=$ awrt 3.4 (months)	dM1	1.1b
		A1	1.1b
		(4)	
(d)	either one of 299 or 300	B1	3.4
		(1)	
			(10 marks)

Notes for Question 14	
14 (b)	
M1:	Attempts to differentiate using • the chain rule to give $\frac{dN}{dt} = \pm Ae^{-0.25t}(3+7e^{-0.25t})^{-2}$ or $\frac{\pm Ae^{-0.25t}}{(3+7e^{-0.25t})^2}$ o.e. • the quotient rule to give $\frac{dN}{dt} = \frac{(3+7e^{-0.25t})(0) \pm Ae^{-0.25t}}{(3+7e^{-0.25t})^2}$ • implicit differentiation to give $N(3+7e^{-0.25t}) = 900 \Rightarrow (3+7e^{-0.25t})\frac{dN}{dt} \pm ANe^{-0.25t} = 0$, o.e. where $A \neq 0$
Note:	Condone a slip in copying $(3+7e^{-0.25t})$ for the M mark
A1:	A correct differentiation statement
Note:	Implicit differentiation gives $(3+7e^{-0.25t})\frac{dN}{dt} - 1.75Ne^{-0.25t} = 0$
dM1:	**Way 1:** Complete attempt, by eliminating *t*, to form an equation linking $\frac{dN}{dt}$ and N only **Way 2:** Complete substitution of $N = \frac{900}{3+7e^{-0.25t}}$ into $\frac{dN}{dt} = \frac{N(300-N)}{1200}$
Note:	**Way 1:** e.g. substitutes $3+7e^{-0.25t} = \frac{900}{N}$ and $e^{-0.25t} = \frac{900}{N}$ or substitutes $e^{-0.25t} = \frac{\frac{900}{N} - 3}{7}$ into their $\frac{dN}{dt} = ...$ to form an equation linking $\frac{dN}{dt}$ and N
A1*:	**Way 1:** Correct algebra leading to $\frac{dN}{dt} = \frac{N(300-N)}{1200}$ * **Way 2:** See scheme
(c)	
B1:	Deduces or shows that $\frac{dN}{dt}$ is maximised when $N = 150$
M1:	Uses the model $N = \frac{900}{3+7e^{-0.25t}}$ with their $N = 150$ and proceeds as far as $e^{-0.25T} = k,\ k > 0$ or $e^{0.25T} = k,\ k > 0$. Condone $t \equiv T$
dM1:	Correct method of using logarithms to find a value for T. Condone $t \equiv T$
A1:	see scheme
Note:	$\frac{d^2N}{dt^2} = \frac{dN}{dt}\left(\frac{300}{1200} - \frac{2N}{1200}\right) = 0 \Rightarrow N = 150$ is acceptable for B1
Note:	Ignore units for T
Note:	Applying $300 = \frac{900}{3+7e^{-0.25t}} \Rightarrow t = ...$ or $0 = \frac{900}{3+7e^{-0.25t}} \Rightarrow t = ...$ is M0 dM0 A0
Note:	M1 dM1 can only be gained in (c) by using an N value in the range $90 < N < 300$
(d)	
B1:	300 (or accept 299)

Question	Scheme	Marks	AOs
14	$N=\dfrac{900}{3+7\mathrm{e}^{-0.25t}}=900(3+7\mathrm{e}^{-0.25t})^{-1},\ t\in\mathbb{R},\ t\geq 0\ ;\quad \dfrac{\mathrm{d}N}{\mathrm{d}t}=\dfrac{N(300-N)}{1200}$		
(b) Way 3	$\displaystyle\int \frac{1}{N(300-N)}\,\mathrm{d}N=\int\frac{1}{1200}\,\mathrm{d}t$	M1	2.1
	$\displaystyle\int \frac{1}{300}\left(\frac{1}{N}+\frac{1}{300-N}\right)\mathrm{d}N=\int\frac{1}{1200}\,\mathrm{d}t$ $\dfrac{1}{300}\ln N-\dfrac{1}{300}\ln(300-N)=\dfrac{1}{1200}t\ \{+c\}$	A1	1.1b
	$\{t=0,\ N=90\Rightarrow\}\ \ c=\dfrac{1}{300}\ln(90)-\dfrac{1}{300}\ln(210)\ \Rightarrow c=\dfrac{1}{300}\ln\left(\dfrac{3}{7}\right)$ $\dfrac{1}{300}\ln N-\dfrac{1}{300}\ln(300-N)=\dfrac{1}{1200}t+\dfrac{1}{300}\ln\left(\dfrac{3}{7}\right)$ $\ln N-\ln(300-N)=\dfrac{1}{4}t+\ln\left(\dfrac{3}{7}\right)$ $\ln\left(\dfrac{N}{300-N}\right)=\dfrac{1}{4}t+\ln\left(\dfrac{3}{7}\right)\ \Rightarrow\ \dfrac{N}{300-N}=\dfrac{3}{7}\mathrm{e}^{\frac{1}{4}t}$	dM1	2.1
	$7N=3\mathrm{e}^{\frac{1}{4}t}(300-N)\ \Rightarrow 7N+3N\mathrm{e}^{\frac{1}{4}t}=900\mathrm{e}^{\frac{1}{4}t}$ $N(7+3\mathrm{e}^{\frac{1}{4}t})=900\mathrm{e}^{\frac{1}{4}t}\ \Rightarrow\ N=\dfrac{900\mathrm{e}^{\frac{1}{4}t}}{7+3\mathrm{e}^{\frac{1}{4}t}}\ \Rightarrow\ N=\dfrac{900}{3+7\mathrm{e}^{-0.25t}}$ *	A1*	1.1b
		(4)	
(b) Way 4	$N(3+7\mathrm{e}^{-0.25t})=900\ \Rightarrow \mathrm{e}^{-0.25t}=\dfrac{1}{7}\left(\dfrac{900}{N}-3\right)\Rightarrow\ \mathrm{e}^{-0.25t}=\dfrac{900-3N}{7N}$	M1	2.1
	$\Rightarrow t=-4\big(\ln(900-3N)-\ln(7N)\big)$ $\Rightarrow\dfrac{\mathrm{d}t}{\mathrm{d}N}=-4\left(\dfrac{-3}{900-3N}-\dfrac{7}{7N}\right)$	A1	1.1b
	$\dfrac{\mathrm{d}t}{\mathrm{d}N}=4\left(\dfrac{1}{300-N}+\dfrac{1}{N}\right)\Rightarrow\ \dfrac{\mathrm{d}t}{\mathrm{d}N}=4\left(\dfrac{N+300-N}{N(300-N)}\right)$	dM1	2.1
	$\dfrac{\mathrm{d}t}{\mathrm{d}N}=\left(\dfrac{1200}{N(300-N)}\right)\Rightarrow\ \dfrac{\mathrm{d}N}{\mathrm{d}t}=\dfrac{N(300-N)}{1200}$ *	A1*	1.1b
		(4)	

Notes for Question 14 Continued	
(b) Way 3	
M1:	Separates the variables, an attempt to form and apply partial fractions and integrates to give $\ln$ terms $= kt\ \{+c\},\ \ k \neq 0$, with or without a constant of integration c
A1:	$\frac{1}{300}\ln N - \frac{1}{300}\ln(300-N) = \frac{1}{1200}t\ \ \{+c\}$ or equivalent with or without a constant of integration c
dM1:	Uses $t=0,\ N=90$ to find their constant of integration and obtains an expression of the form $\lambda e^{\frac{1}{4}t} = f(N);\ \lambda \neq 0$ or $\lambda e^{-\frac{1}{4}t} = f(N);\ \lambda \neq 0$
A1*:	Correct manipulation leading to $N = \frac{900}{3+7e^{-0.25t}}$ *
(b) Way 4	
M1:	Valid attempt to make t the subject, followed by an attempt to find two ln derivatives, condoning sign errors and constant errors.
A1:	$\frac{dt}{dN} = -4\left(\frac{-3}{900-3N} - \frac{7}{7N}\right)$ or equivalent
dM1:	Forms a common denominator to combine their fractions
A1*:	Correct algebra leading to $\frac{dN}{dt} = \frac{N(300-N)}{1200}$ *

3. $$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(a) Show that $\frac{dy}{dx} = \frac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(b) Hence deduce the range of values for x for which $\frac{dy}{dx} < 0$

(1)

Question	Scheme	Marks	AOs
3 (a)	Correct method used in attempting to differentiate $y=\dfrac{5x^2+10x}{(x+1)^2}$	M1	3.1a
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+1)^2\times(10x+10)-(5x^2+10x)\times2(x+1)}{(x+1)^4}$ oe	A1	1.1b
	Factorises/Cancels term in $(x+1)$ and attempts to simplify $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+1)\times(10x+10)-(5x^2+10x)\times2}{(x+1)^3}=\dfrac{A}{(x+1)^3}$	M1	2.1
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{10}{(x+1)^3}$	A1	1.1b
		(4)	
(b)	For $x<-1$ Follow through on their $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{A}{(x+1)^n}, n=1,3$	B1ft	2.2a
		(1)	
			(5 marks)

(a)

M1: Attempts to use a correct rule to differentiate Eg: Use of quotient (& chain) rules on $y=\dfrac{5x^2+10x}{(x+1)^2}$

Alternatively uses the product (and chain) rules on $y=(5x^2+10x)(x+1)^{-2}$

Condone slips but expect $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\dfrac{(x+1)^2\times(Ax+B)-(5x^2+10x)\times(Cx+D)}{(x+1)^4}$ $(A,B,C,D>0)$ or

$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\dfrac{(x+1)^2\times(Ax+B)-(5x^2+10x)\times(Cx+D)}{\left((x+1)^2\right)^2}$ $(A,B,C,D>0)$ using the quotient rule

or $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=(x+1)^{-2}\times(Ax+B)+(5x^2+10x)\times C(x+1)^{-3}$ $(A,B,C\neq0)$ using the product rule.

Condone missing brackets and slips for the M mark. For instance if they quote $u=5x^2+10$, $v=(x+1)^2$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow where they quote the correct formula, give values of u and v, but only have v rather than v^2 the denominator.

A1: A correct (unsimplified) answer

Eg. $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{(x+1)^2\times(10x+10)-(5x^2+10x)\times 2(x+1)}{(x+1)^4}$ or equivalent via the quotient rule.

OR $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=(x+1)^{-2}\times(10x+10)+(5x^2+10x)\times -2(x+1)^{-3}$ or equivalent via the product rule

M1: A valid attempt to proceed to the given form of the answer.

It is dependent upon having a quotient rule of $\pm\frac{v\mathrm{d}u-u\mathrm{d}v}{v^2}$ and proceeding to $\frac{A}{(x+1)^3}$

It can also be scored on a quotient rule of $\pm\frac{v\mathrm{d}u-u\mathrm{d}v}{v}$ and proceeding to $\frac{A}{(x+1)}$

You may see candidates expanding terms in the numerator. FYI $10x^3+30x^2+30x+10-10x^3-30x^2-20x$ but under this method they must reach the same expression as required by the main method.

Using the product rule expect to see a common denominator being used correctly before the above

A1: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{10}{(x+1)^3}$ There is no requirement to see $\frac{\mathrm{d}y}{\mathrm{d}x}=$ and they can recover from missing brackets/slips.

(b)

B1ft: Score for deducing the correct answer of $x<-1$ This can be scored independent of their answer to part (a). Alternatively score for a correct **ft** answer for their $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{A}{(x+1)^n}$ where $A<0$ and $n=1,3$ award for $x>-1$. So for example if $A>0$ and $n=1,3\Rightarrow x<-1$

Question	Scheme	Marks	AOs
Alt via division	Writes $y=\frac{5x^2+10x}{(x+1)^2}$ in form $y=A\pm\frac{B}{(x+1)^2}$ $A,B\neq 0$	M1	3.1a
	Writes $y=\frac{5x^2+10x}{(x+1)^2}$ in the form $y=5-\frac{5}{(x+1)^2}$	A1	1.1b
	Uses the chain rule $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{C}{(x+1)^3}$ (May be scored from $A=0$)	M1	2.1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{10}{(x+1)^3}$ which cannot be awarded from incorrect value of A	A1	1.1b
		(4)	
(b)	For $x<-1$ or correct follow through	B1ft	2.2a
		(1)	
		(5 marks)	

Question	Scheme	Marks	AOs

12. $$f(x) = 10e^{-0.25x} \sin x, \qquad x \geqslant 0$$

(a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

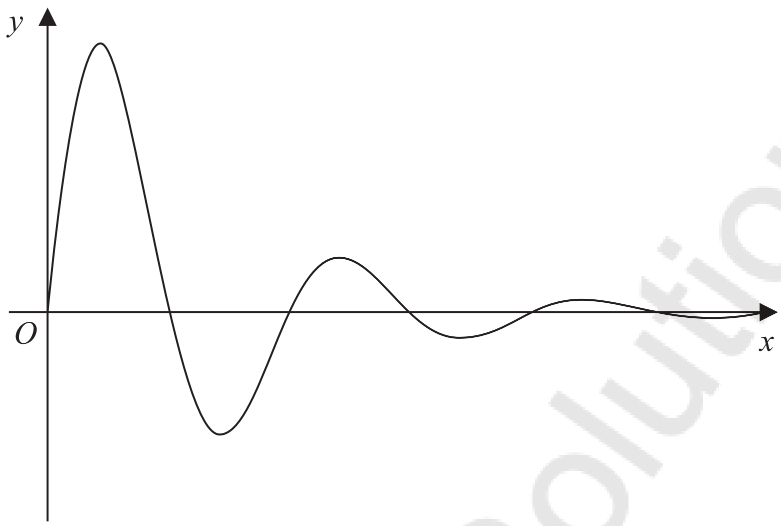


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

(b) Sketch the graph of H against t where

$$H(t) = \left|10e^{-0.25t} \sin t\right| \qquad t \geqslant 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

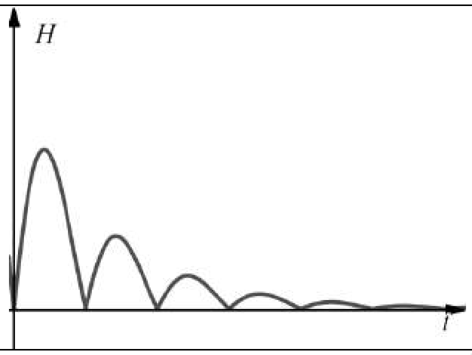
Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce.

(3)

(d) Explain why this model should not be used to predict the time of each bounce.

(1)

Question	Scheme	Marks	AOs
12 (a)	$\mathrm{f}(x) = 10\mathrm{e}^{-0.25x}\sin x$		
	$\Rightarrow \mathrm{f}'(x) = -2.5\mathrm{e}^{-0.25x}\sin x + 10\mathrm{e}^{-0.25x}\cos x$ oe	M1 A1	1.1b 1.1b
	$\mathrm{f}'(x) = 0 \Rightarrow -2.5\cancel{\mathrm{e}^{-0.25x}}\sin x + 10\cancel{\mathrm{e}^{-0.25x}}\cos x = 0$	M1	2.1
	$\frac{\sin x}{\cos x} = \frac{10}{2.5} \Rightarrow \tan x = 4*$	A1*	1.1b
		(4)	
(b)	 "Correct" shape for 2 loops Fully correct with decreasing heights	M1 A1	1.1b 1.1b
		(2)	
(c)	Solves $\tan x = 4$ and substitutes answer into $H(t)$	M1	3.1a
	$H(4.47) = \left\lvert 10\mathrm{e}^{-0.25\times 4.47}\sin 4.47 \right\rvert$	M1	1.1b
	awrt 3.18 (metres)	A1	3.2a
		(3)	
(d)	The **times between** each bounce should not stay the same when the heights of each bounce is getting smaller	B1	3.5b
		(1)	
			(10 marks)

(a)

M1: For attempting to differentiate using the product rule condoning slips, for example the power of e .

So for example score expressions of the form $\pm ...\mathrm{e}^{-0.25x}\sin x \pm ...\mathrm{e}^{-0.25x}\cos x$ M1

Sight of $vdu - udv$ however is M0

A1: $\mathrm{f}'(x) = -2.5\mathrm{e}^{-0.25x}\sin x + 10\mathrm{e}^{-0.25x}\cos x$ which may be unsimplified

M1: For clear reasoning in setting their $\mathrm{f}'(x) = 0,$ factorising/ cancelling out the $\mathrm{e}^{-0.25x}$ term leading to a trigonometric equation in only $\sin x$ and $\cos x$

Do not allow candidates to substitute $x = \arctan 4$ into $\mathrm{f}'(x)$ to score this mark.

A1*: Shows the steps $\frac{\sin x}{\cos x} = \frac{10}{2.5}$ or equivalent leading to $\Rightarrow \tan x = 4*$. $\frac{\sin x}{\cos x}$ must be seen.

(b)

M1: Draws at least two "loops". The height of the second loop should be lower than the first loop.

Condone the sight of rounding where there should be cusps

A1: At least 4 loops with decreasing heights and no rounding at the cusps.

The intention should be that the graph should ‘sit’ on the x -axis but be tolerant.

It is possible to overwrite Figure 3, but all loops must be clearly seen.

(c)

M1: Understands that to solve the problem they are required to substitute an answer to $\tan t = 4$ into $H(t)$

This can be awarded for an attempt to substitute $t =$ awrt 1.33 or $t =$ awrt 4.47 into $H(t)$

$H(t) = 6.96$ implies the use of $t = 1.33$ Condone for this mark only, an attempt to substitute $t =$ awrt 76° or awrt 256° into $H(t)$

M1: Substitutes $t =$ awrt 4.47 into $H(t) = \left|10\mathrm{e}^{-0.25t} \sin t\right|$. Implied by awrt 3.2

A1: Awrt 3.18 metres. Condone the lack of units. If two values are given the correct one must be seen to have been chosen

It is possible for candidates to sketch this on their graphical calculators and gain this answer. If there is no incorrect working seen and 3.18 is given, then award 111 for such an attempt.

(d)

B1: Makes reference to the fact that the time between each bounce should not stay the same when the heights of each bounce is getting smaller.

Look for " time (or gap) between the bounces will change"

'bounces would not be equal times apart'

'bounces would become more frequent'

But do not accept 'the times between each bounce would be longer or slower'

Do not accept explanations such as there are other factors that would affect this such as "wind resistance", friction etc

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{dy}{dx}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{dy}{dx} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

Question	Scheme	Marks	AOs
14 (a)	Attempts to differentiate $x = 4\sin 2y$ and inverts $\frac{dx}{dy} = 8\cos 2y \Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 2y}$	M1	1.1b
	At (0,0) $\frac{dy}{dx} = \frac{1}{8}$	A1	1.1b
		(2)	
(b)	(i) Uses $\sin 2y \approx 2y$ when y is small to obtain $x \approx 8y$	B1	1.1b
	(ii) The value found in (a) is the gradient of the line found in (b)(i)	B1	2.4
		(2)	
(c)	Uses their $\frac{dy}{dx}$ as a function of y and, using both $\sin^2 2y + \cos^2 2y = 1$ and $x = 4\sin 2y$ in an attempt to write $\frac{dy}{dx}$ or $\frac{dx}{dy}$ as a function of x Allow for $\frac{dy}{dx} = k\frac{1}{\cos 2y} = ..\frac{1}{\sqrt{1-(..x)^2}}$	M1	2.1
	A correct answer $\frac{dy}{dx} = \frac{1}{8\sqrt{1-\left(\frac{x}{4}\right)^2}}$ or $\frac{dx}{dy} = 8\sqrt{1-\left(\frac{x}{4}\right)^2}$	A1	1.1b
	and in the correct form $\frac{dy}{dx} = \frac{1}{2\sqrt{16-x^2}}$	A1	1.1b
		(3)	
			(7 marks)

(a)

M1: Attempts to differentiate $x = 4\sin 2y$ and inverts.

Allow for $\frac{dx}{dy} = k\cos 2y \Rightarrow \frac{dy}{dx} = \frac{1}{k\cos 2y}$ or $1 = k\cos 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{k\cos 2y}$

Alternatively, changes the subject and differentiates $x = 4\sin 2y \rightarrow y = ...\arcsin\left(\frac{x}{4}\right) \rightarrow \frac{dy}{dx} = \frac{...}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$

It is possible to approach this from $x = 8\sin y\cos y \Rightarrow \frac{dx}{dy} = \pm 8\sin^2 y \pm 8\cos^2 y$ before inverting

A1: $\frac{dy}{dx} = \frac{1}{8}$ Allow both marks for sight of this answer as long as no incorrect working is seen (See below)

Watch for candidates who reach this answer via $\frac{dx}{dy} = 8\cos 2x \Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 2x}$ This is M0 A0

(b)(i)

B1: Uses $\sin 2y \approx 2y$ when y is small to obtain $x = 8y$ oe such as $x = 4(2y)$.

Do not allow $\sin 2y \approx 2\theta$ to get $x = 8\theta$ but allow recovery in (b)(i) or (b)(ii)

Double angle formula is B0 as it does not satisfy the demands of the question.

(b)(ii)

B1: Explains the relationship between the answers to (a) and (b) (i).

For this to be scored the first three marks, in almost all cases, must have been awarded and the statement must refer to both answers

Allow for example "The gradients are the same $\left(=\frac{1}{8}\right)$" 'both have $m = \frac{1}{8}$'

Do not accept the statement that 8 and $\frac{1}{8}$ are reciprocals of each other unless further correct work explains the relationship in terms of $\frac{dx}{dy}$ and $\frac{dy}{dx}$

(c)

M1: Uses their $\frac{dy}{dx}$ as a function of y and, using both $\sin^2 2y + \cos^2 2y = 1$ and $x = 4\sin 2y$, attempts to write $\frac{dy}{dx}$ or $\frac{dx}{dy}$ as a function of x. The $\frac{dy}{dx}$ may not be seen and may be implied by their calculation.

A1: A correct (un-simplified) answer for $\frac{dy}{dx}$ or $\frac{dx}{dy}$ Eg. $\frac{dy}{dx} = \frac{1}{8\sqrt{1-\left(\frac{x}{4}\right)^2}}$

A1: $\frac{dy}{dx} = \frac{1}{2\sqrt{16-x^2}}$ The $\frac{dy}{dx}$ must be seen at least once in part (c) of this solution

....

Alt to (c) using arcsin

M1: Alternatively, changes the subject and differentiates $x = 4\sin 2y \rightarrow y = ...\arcsin\left(\frac{x}{4}\right) \rightarrow \frac{dy}{dx} = \frac{...}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$

Condone a lack of bracketing on the $\frac{x}{4}$ which may appear as $\frac{x^2}{4}$

A1: $\frac{dy}{dx} = \frac{1/8}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$ oe

A1: $\frac{dy}{dx} = \frac{1}{2\sqrt{16-x^2}}$

9.

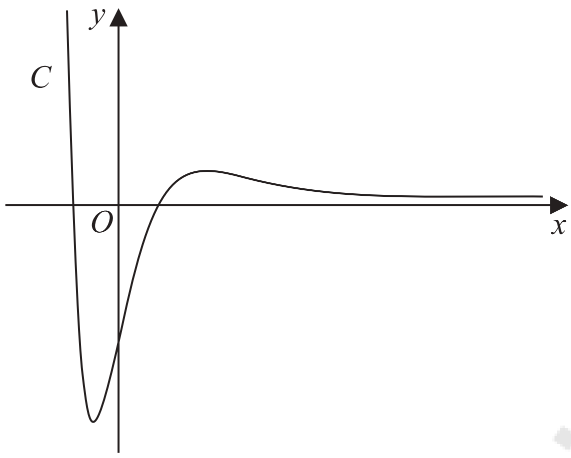


Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

(a) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)\mathrm{e}^{-2x}$

(3)

(b) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

The function g and the function h are defined by

$$\mathrm{g}(x) = 2\mathrm{f}(x) \qquad x \in \mathbb{R}$$

$$\mathrm{h}(x) = 2\mathrm{f}(x) - 3 \qquad x \geqslant 0$$

(c) Find (i) the range of g

(ii) the range of h

(3)

Question	Scheme	Marks	AOs
9(a)	$f(x)=4\left(x^2-2\right)e^{-2x}$		
	Differentiates to $e^{-2x}\times 8x+4\left(x^2-2\right)\times -2e^{-2x}$	M1 A1	1.1b 1.1b
	$f'(x)=8e^{-2x}\left\{x-\left(x^2-2\right)\right\}=8\left(2+x-x^2\right)e^{-2x}$ *	A1*	2.1
		(3)	
(b)	States roots of $f'(x)=0$ $x=-1,2$	B1	1.1b
	Substitutes one x value to find a y value	M1	1.1b
	Stationary points are $\left(-1,-4e^2\right)$ and $\left(2,8e^{-4}\right)$	A1	1.1b
		(3)	
(c)	(i) Range $\left[-8e^2,\infty\right)$ o.e. such as $g(x)\geqslant -8e^2$	B1ft	2.5
	(ii) For • Either attempting to find $2f(0)-3=2\times -8-3=(-19)$ and identifying this as the **lower bound** • Or attempting to find $2\times$"$8e^{-4}$"-3 and identifying this as the **upper bound**	M1	3.1a
	Range $\left[-19,16e^{-4}-3\right]$	A1	1.1b
		(3)	
			(9 marks)
Notes:			

(a)

M1: Attempts the product rule and uses $e^{-2x}\to ke^{-2x},\quad k\neq 0$

If candidate states $u=4(x^2-2), v=e^{-2x}$ with $u'=....,v'=...e^{-2x}$ it can be implied by their $vu'+uv'$

If they just write down an answer without working award for $f'(x)=pxe^{-2x}\pm q\left(x^2-2\right)e^{-2x}$

They may multiply out first $f(x)=4x^2e^{-2x}-8e^{-2x}$. Apply in the same way condoning slips

Alternatively attempts the quotient rule on $f(x)=\frac{u}{v}=\frac{4\left(x^2-2\right)}{e^{2x}}$ with $v'=ke^{2x}$ and $f'(x)=\frac{vu'-uv'}{v^2}$

A1: A correct $f'(x)$ which may be unsimplified.

Via the quotient rule you can award for $f'(x)=\frac{8xe^{2x}-8\left(x^2-2\right)e^{2x}}{e^{4x}}$ o.e.

A1*: Proceeds correctly to given answer showing all necessary steps.

The $f'(x)$ or $\frac{dy}{dx}$ must be present at some point in the solution

This is a "show that" question and there must not be any errors. All bracketing must be correct.

Allow a candidate to move from the **simplified** unfactorised answer of $f'(x)=8xe^{-2x}-8\left(x^2-2\right)e^{-2x}$

to the given answer in one step.

Do not allow it from an **unsimplified** $f'(x) = 4 \times 2xe^{-2x} + 4(x^2 - 2) \times -2e^{-2x}$

Allow the expression / bracketed expression to be written in a different order.

So, for example, $8(x - x^2 + 2)e^{-2x}$ is OK

(b)

B1: States or implies $x = -1, 2$ (as the roots of $f'(x) = 0$)

M1: Substitutes one x value of their solution to $f'(x) = 0$ in f(x) to find a y value.

Allow decimals here (3sf). FYI, to 3 sf, $-4e^2 = -29.6$ and $8e^{-4} = 0.147$

Some candidates just write down the x coordinates but then go on in part (c) to find the ranges using the y coordinates. Allow this mark to be scored from work in part (c)

A1: Obtains $(-1, -4e^2)$ and $(2, 8e^{-4})$ as the stationary points. This must be scored in (b). Remember to isw after a correct answer. Allow these to be written separately. E.g. $x = -1, y = -4e^2$

Extra solutions, e.g. from $x = 0$ will be penalised on this mark.

(c)(i)

B1ft: For a correct range written using correct notation.

Follow through on 2 × their minimum "y" value from part (b), providing it is negative.

Condone a decimal answer if this is consistent with their answer in (b) to 3sf or better.

Examples of correct responses are $[-8e^2, \infty)$, $g \geqslant -8e^2$, $y \geqslant -8e^2$, $\{q \in \mathbb{R}, q \geqslant -8e^2\}$

(c)(ii)

M1: See main scheme. Follow through on $2 \times$ their "$8e^{-4}$" $- 3$ for the upper bound.

A1: Range $[-19, 16e^{-4} - 3]$ o.e. such as $-19 \leqslant y \leqslant 16e^{-4} - 3$ but must be exact

15. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

Question	Scheme	Marks	AOs
15 (a)	$x^2 \tan y = 9 \Rightarrow 2x\tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1+\tan^2 y = 1 + f(x)$	M1	1.1b
	$\frac{dy}{dx} = \frac{-2x \times \frac{9}{x^2}}{x^2\left(1+\frac{81}{x^4}\right)} = \frac{-18x}{x^4+81}$ *	A1*	2.1
		(4)	
(b)	$\frac{dy}{dx} = \frac{-18x}{x^4+81}$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{-18\times\left(x^4+81\right)-\left(-18x\right)\left(4x^3\right)}{\left(x^4+81\right)^2} = \frac{54\left(x^4-27\right)}{\left(x^4+81\right)^2}$ o.e.	M1 A1	1.1b 1.1b
	States that when $x < \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} < 0$ when $x = \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} = 0$ **AND** when $x > \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} > 0$ giving a point of inflection when $x = \sqrt[4]{27}$	A1	2.4
		(3)	
			(7 marks)
Notes:			

(a)

M1: Attempts to differentiate $\tan y$ implicitly. Eg. $\tan y \to \sec^2 y \frac{dy}{dx}$ or $\cot y \to -\text{cosec}^2 y \frac{dy}{dx}$

You may well see an attempt $\tan y = \frac{9}{x^2} \Rightarrow \sec^2 y \frac{dy}{dx} = ...$

When a candidate writes $x^2 \tan y = 9 \Rightarrow x = 3\tan^{-\frac{1}{2}} y$ the mark is scored for $\tan^{-\frac{1}{2}} y \to ...\tan^{-\frac{3}{2}} y \sec^2 y$

A1: Correct differentiation $2x\tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$

Allow also $\sec^2 y \frac{dy}{dx} = -\frac{18}{x^3}$ or $2x = -9\text{cosec}^2 y \frac{dy}{dx}$ amongst others

M1: Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1+\tan^2 y = 1+f(x)$

A1*: Proceeds correctly to the given answer of $\frac{dy}{dx} = \frac{-18x}{x^4+81}$

(b)

M1: Attempts to differentiate the given expression using the product or quotient rule.

For example look for a correct attempt at $\frac{vu'-uv'}{v^2}$ with $u=-18x, v=x^4+81, u'=\pm18, v'=...x^3$

If no method is seen or implied award for $\frac{\pm18\times\left(x^4+81\right)\pm18x\left(ax^3\right)}{\left(x^4+81\right)^2}$

Using the product rule award for $\pm18\left(x^4+81\right)^{-1}\pm18x\left(x^4+81\right)^{-2}\times cx^3$

A1: Correct **simplified** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{54\left(x^4-27\right)}{\left(x^4+81\right)^2}$ o.e. such as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{54x^4-1458}{\left(x^4+81\right)^2}$

Alternatively score for showing that when a correct (unsimplified) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0\Rightarrow x^4=27\Rightarrow x=\sqrt[4]{27}$

Or for substituting $x=\sqrt[4]{27}$ into an unsimplified but correct $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and showing that it is 0

A1: Correct explanation with a minimal conclusion and correct second derivative.
See scheme.

It can be also be argued from $x^4<27$, $x^4=27$ and $x^4>27$ provided the conclusion states that the point of inflection is at $x=\sqrt[4]{27}$

Alternatively substitutes values of x either side of $\sqrt[4]{27}$ and at $\sqrt[4]{27}$, into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, finds all three values and makes a minimal conclusion.

A different method involves finding $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and showing that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\neq0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ when $x=\sqrt[4]{27}$

FYI $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{23328x^3}{\left(x^4+81\right)^3}=0.219$ when $x=\sqrt[4]{27}$

Alternative part (a) using arctan

M1: Sets $y=\arctan\frac{9}{x^2}\rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times...$ where ... could be 1

A2: $y=\arctan\frac{9}{x^2}\rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times-\frac{18}{x^3}$

A1*: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\frac{81}{x^4}}\times-\frac{18}{x^3}=\frac{-18x}{x^4+1}$ showing correct intermediate step and no errors.

13. The function g is defined by

$$g(x) = \frac{3\ln(x) - 7}{\ln(x) - 2} \qquad x > 0 \qquad x \neq k$$

where k is a constant.

(a) Deduce the value of k.

(1)

(b) Prove that

$$g'(x) > 0$$

for all values of x in the domain of g.

(3)

(c) Find the range of values of a for which

$$g(a) > 0$$

(2)

Question	Scheme	Marks	AOs
13(a)	$k = e^2$ **or** $x \neq e^2$	B1	2.2a
		(1)	
(b)	$g'(x) = \dfrac{(\ln x - 2)\times\dfrac{3}{x} - (3\ln x - 7)\times\dfrac{1}{x}}{(\ln x - 2)^2} = \dfrac{1}{x(\ln x - 2)^2}$ or $g'(x) = \dfrac{d}{dx}\left(3 - (\ln(x) - 2)^{-1}\right) = (\ln x - 2)^{-2}\times\dfrac{1}{x} = \dfrac{1}{x(\ln x - 2)^2}$ or $g'(x) = (\ln x - 2)^{-1}\times\dfrac{3}{x} - (3\ln x - 7)(\ln x - 2)^{-2}\times\dfrac{1}{x} = \dfrac{1}{x(\ln x - 2)^2}$	M1 A1	1.1b 2.1
	As $x > 0$ (or $1/x > 0$) **AND** $\ln x - 2$ is squared so $g'(x) > 0$	A1cso	2.4
		(3)	
(c)	Attempts to solve either $3\ln x - 7 \ldots 0$ or $\ln x - 2 \ldots 0$ or $3\ln a - 7 \ldots 0$ or $\ln a - 2 \ldots 0$ where … is "=" or ">" to reach a value for x or a but may be seen as an inequality e.g. $x > \ldots$ or $a > \ldots$	M1	3.1a
	$0 < a < e^2, \quad a > e^{\frac{7}{3}}$	A1	2.2a
		(2)	
			(6 marks)

Notes:

(a)

B1: Deduces $k = e^2$ **or** $x \neq e^2$ Condone k = awrt 7.39 or $x \neq$ awrt 7.39

(b)

M1: Attempts to differentiate via the quotient rule and with $\ln x \to \dfrac{1}{x}$ so allow for:

$$\frac{d}{dx}(g(x)) = \frac{(\ln x - 2)\times\frac{\alpha}{x} - (3\ln x - 7)\times\frac{\beta}{x}}{(\ln x - 2)^2}, \ \beta > 0$$

But a correct rule may be implied by their *u*, *v*, *u'*, *v'* followed by applying $\dfrac{vu' - uv'}{v^2}$ etc.

Alternatively attempts to write $g(x) = \dfrac{3\ln(x) - 7}{\ln(x) - 2} = 3 - (\ln(x) - 2)^{-1}$ and attempts the chain rule so allow for:

$$3 - (\ln(x) - 2)^{-1} \to (\ln(x) - 2)^{-2}\times\frac{\alpha}{x}$$

Alternatively writes $g(x) = (3\ln(x) - 7)(\ln(x) - 2)^{-1}$ and attempts the product rule so allow for:

$$g'(x) = (\ln x - 2)^{-1}\times\frac{\alpha}{x} - (3\ln x - 7)(\ln x - 2)^{-2}\times\frac{\beta}{x}$$

In general condone missing brackets for the M mark. E.g. if they quote $u = 3\ln x - 7$ and $v = \ln x - 2$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow this mark if they quote the correct quotient rule but only have v rather than v^2 in the denominator.

A1: $\dfrac{1}{x(\ln x-2)^2}$ **Allow** $\dfrac{\frac{1}{x}}{(\ln x-2)^2}$ **i.e. we need to see the numerator simplified to 1/*x***

Note that some candidates establish the correct numerator and correct denominator independently and provided they obtain the correct expressions, this mark can be awarded.

But allow a correctly expanded denominator.

A1cso: States that as $x > 0$ **AND** $\ln x - 2$ is squared so $g'(x) > 0$

(c)

M1: Attempts to solve either $3\ln x - 7 = 0$ or $\ln x - 2 = 0$ or using inequalities e.g. $3\ln x - 7 > 0$

A1: $0 < a < e^2, \ a > e^{\frac{7}{3}}$

14. Given that

$$y = \frac{x - 4}{2 + \sqrt{x}} \qquad x > 0$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

Question	Scheme	Marks	AOs
14	$y=\frac{x-4}{2+\sqrt{x}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2+\sqrt{x}-(x-4)\frac{1}{2}x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}$	M1 A1	2.1 1.1b
	$=\frac{2+\sqrt{x}-(x-4)\frac{1}{2}x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}=\frac{2+\sqrt{x}-\frac{1}{2}\sqrt{x}+2x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}=\frac{2\sqrt{x}+\frac{1}{2}x+2}{\sqrt{x}\left(2+\sqrt{x}\right)^2}$	M1	1.1b
	$=\frac{x+4\sqrt{x}+4}{2\sqrt{x}\left(2+\sqrt{x}\right)^2}=\frac{\left(2+\sqrt{x}\right)^2}{2\sqrt{x}\left(2+\sqrt{x}\right)^2}=\frac{1}{2\sqrt{x}}$	A1	2.1
		(4)	
		(4 marks)	

Notes

M1: Attempts to use a correct rule e.g. quotient or product (& chain) rule to achieve the following forms

Quotient : $\frac{\alpha\left(2+\sqrt{x}\right)-\beta(x-4)x^{-\frac{1}{2}}}{\left(2+\sqrt{x}\right)^2}$ but be tolerant of attempts where the $\left(2+\sqrt{x}\right)^2$ has been incorrectly expanded

Product: $\alpha\left(2+\sqrt{x}\right)^{-1}+\beta x^{-\frac{1}{2}}(x-4)\left(2+\sqrt{x}\right)^{-2}$

Alternatively with $t=\sqrt{x}$, $y=\frac{t^2-4}{2+t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{2t(2+t)-\left(t^2-4\right)}{(2+t)^2}\times\frac{1}{2}x^{-\frac{1}{2}}$ with same rules

A1: Correct derivative in any form. Must be in terms of a single variable (which could be t)

M1: Following a correct attempt at differentiation, it is scored for multiplying both numerator and denominator by $\sqrt{x}$ and collecting terms to form a single fraction. It can also be scored from $\frac{uv'-vu'}{v^2}$

For the $t=\sqrt{x}$, look for an attempt to simplify $\frac{t^2+4t+4}{(2+t)^2}\times\frac{1}{2t}$

A1: Correct expression showing all key steps with no errors or omissions. $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be seen at least once

Question	Scheme	Marks	AOs
14	$y=\frac{x-4}{2+\sqrt{x}} \Rightarrow y=\frac{\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)}{2+\sqrt{x}}=\sqrt{x}-2$	M1 A1	2.1 1.1b
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$	M1 A1	1.1b 2.1
		(4)	
		(4 marks)	

Notes

M1: Attempts to use difference of two squares. Can also be scored using

$t=\sqrt{x} \Rightarrow y=\frac{t^2-4}{t+2} \Rightarrow y=\frac{(t+2)(t-2)}{t+2}$

A1: $y=\sqrt{x}-2$ or $y=t-2$

M1: Attempts to differentiate an expression of the form $y=\sqrt{x}+b$

A1: Correct expression showing all key steps with no errors or omissions. $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be seen at least once

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

Question	Scheme	Marks	AOs
8(a)	$\frac{\mathrm{d}}{\mathrm{d}x}(3y^2)=6y\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}}{\mathrm{d}x}(qxy)=qx\frac{\mathrm{d}y}{\mathrm{d}x}+qy$	M1	2.1
	$3px^2+qx\frac{\mathrm{d}y}{\mathrm{d}x}+qy+6y\frac{\mathrm{d}y}{\mathrm{d}x}=0$	A1	1.1b
	$(qx+6y)\frac{\mathrm{d}y}{\mathrm{d}x}=-3px^2-qy\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots$	dM1	2.1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3px^2-qy}{qx+6y}$	A1	1.1b
		(4)	
(b)	$p(-1)^3+q(-1)(-4)+3(-4)^2=26$	M1	1.1b
	$19x+26y+123=0\Rightarrow m=-\frac{19}{26}$	B1	2.2a
	$\frac{-3p(-1)^2-q(-4)}{q(-1)+6(-4)}=\frac{26}{19}$ or $\frac{q(-1)+6(-4)}{3p(-1)^2+q(-4)}=-\frac{19}{26}$	M1	3.1a
	$p-4q=22,\ 57p-102q=624\Rightarrow p=\ldots,q=\ldots$	dM1	1.1b
	$p=2,\ q=-5$	A1	1.1b
		(5)	
		(9 marks)	

Notes

(a)

M1: For selecting the appropriate method of differentiating:

Allow this mark for either $3y^2\to\alpha y\frac{\mathrm{d}y}{\mathrm{d}x}$ or $qxy\to\alpha x\frac{\mathrm{d}y}{\mathrm{d}x}+\beta y$

A1: Fully correct differentiation. Ignore any spurious $\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots$

dM1: A valid attempt to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject with 2 terms only in $\frac{\mathrm{d}y}{\mathrm{d}x}$ coming from qxy and $3y^2$

Depends on the first method mark.

A1: Fully correct expression

(b)

M1: Uses $x=-1$ and $y=-4$ in the equation of C to obtain an equation in p and q

B1: Deduces the correct gradient of the given normal.

This may be implied by e.g.

$19x+26y+123=0\Rightarrow y=-\frac{19}{26}x+\ldots\Rightarrow$ Tangent equation is $y=\frac{26}{19}x+\ldots$

M1: Fully correct strategy to establish an equation connecting p and q using $x=-1$ and $y=-4$ in their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the gradient of the normal. E.g. $(a)=-1\div\text{their}-\frac{19}{26}$ or $-1\div(a)=\text{their}-\frac{19}{26}$

dM1: Solves simultaneously to obtain values for p and q.

Depends on both previous method marks.

A1: Correct values

Alternative for (b):

$$\frac{dy}{dx}=\frac{-3p+4q}{-q-24}\Rightarrow y+4=\frac{q+24}{4q-3p}(x+1)$$

M1A1

$$\Rightarrow y(4q-3p)+4(4q-3p)=(q+24)x+q+24$$

M1

$$19x+26y+123=0\Rightarrow q+24=19\Rightarrow q=-5$$

$$3p-4q=26\Rightarrow 3p+20=26\Rightarrow p=2$$

M1A1

M1: Uses (–1, –4) in the tangent gradient and attempts to form normal equation
A1: Correct equation for normal
M1: Multiplies up so that coefficients can be compared
dM1: Full method comparing coefficients to find values for p and q
A1: Correct values

13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

Question	Scheme	Marks	AOs
13(a)	$y=\text{cosec}^3\theta \Rightarrow \frac{\text{d}y}{\text{d}\theta}=-3\text{cosec}^2\theta\text{cosec}\,\theta\cot\theta$	B1	1.1b
	$\frac{\text{d}y}{\text{d}x}=\frac{\text{d}y}{\text{d}\theta}\div\frac{\text{d}x}{\text{d}\theta}$	M1	1.1b
	$\frac{\text{d}y}{\text{d}x}=\frac{-3\text{cosec}^3\theta\cot\theta}{2\cos 2\theta}$	A1	1.1b
		(3)	
(b)	$y=8\Rightarrow \text{cosec}^3\theta=8\Rightarrow \sin^3\theta=\frac{1}{8}\Rightarrow \sin\theta=\frac{1}{2}$	M1	3.1a
	$\theta=\frac{\pi}{6}\Rightarrow\frac{\text{d}y}{\text{d}x}=\frac{-3\text{cosec}^3\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{2\pi}{6}\right)}=...$ **or** $\sin\theta=\frac{1}{2}\Rightarrow\frac{\text{d}y}{\text{d}x}=\frac{\frac{-3}{\sin^3\theta}\times\frac{\cos\theta}{\sin\theta}}{2\left(1-2\sin^2\theta\right)}=\frac{-3\times8\times\frac{\sqrt{3}/2}{1/2}}{2\left(1-2\times\frac{1}{4}\right)}$	M1	2.1
	$=-24\sqrt{3}$	A1	2.2a
		(3)	
		(6 marks)	

Notes

(a)

B1: Correct expression for $\frac{\text{d}y}{\text{d}\theta}$ seen or implied in any form e.g. $\frac{-3\cos\theta}{\sin^4\theta}$

M1: Obtains $\frac{\text{d}x}{\text{d}\theta}=k\cos 2\theta$ or $\alpha\cos^2\theta+\beta\sin^2\theta$ (from product rule on sinθcosθ)

and attempts $\frac{\text{d}y}{\text{d}x}=\frac{\text{d}y}{\text{d}\theta}\div\frac{\text{d}x}{\text{d}\theta}$

A1: Correct expression in any form.

May see e.g. $\frac{-3\cos\theta}{2\sin^4\theta\cos 2\theta}$, $-\frac{3}{4\sin^4\theta\cos\theta-2\sin^3\theta\tan\theta}$

(b)

M1: Recognises the need to find the value of $\sin\theta$ or θ when y = 8 and uses the y parameter to establish its value. This should be correct work leading to $\sin\theta=\frac{1}{2}$ or e.g. $\theta=\frac{\pi}{6}$ or 30°.

M1: Uses their value of $\sin\theta$ or θ in their $\frac{\text{d}y}{\text{d}x}$ from part (a) (working in exact form) in an attempt to obtain an exact value for $\frac{\text{d}y}{\text{d}x}$. May be implied by a correct exact answer.

If no working is shown but an exact answer is given you may need to check that this follows their $\frac{\text{d}y}{\text{d}x}$.

A1: Deduces the correct gradient

15.

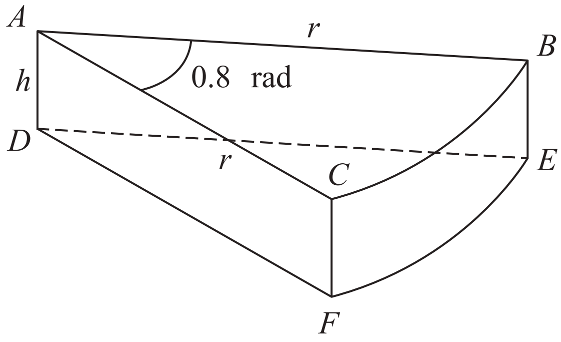


Figure 5

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD, CF and BE are perpendicular to faces ABC and DEF
- edges AD, CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

(a) show that the surface area of the toy, $S\text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

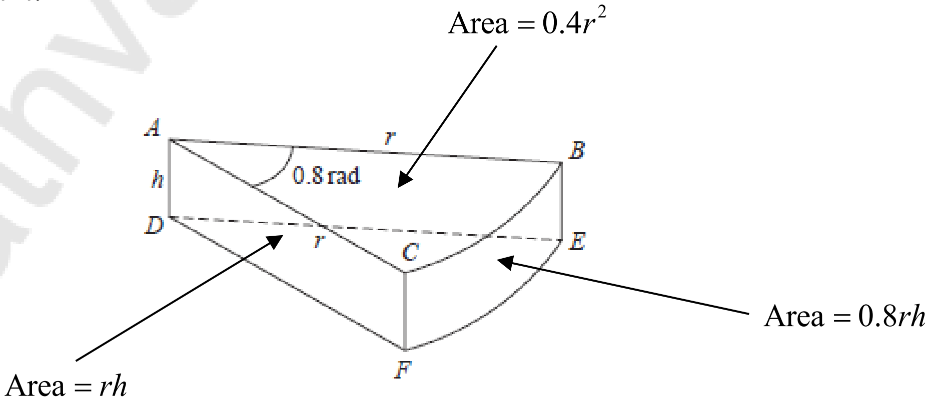
(2)

P 6 9 6 0 1 A 0 4 2 4 8

Question	Scheme	Marks	AOs
15 (a)	Sets up an allowable equation using volume $=240$ E.g. $\frac{1}{2}r^2\times 0.8h=240\Rightarrow h=\frac{600}{r^2}$ o.e.	M1 A1	3.4 1.1b
	Attempts to substitute their $h=\frac{600}{r^2}$ into $(S=)\frac{1}{2}r^2\times 0.8+\frac{1}{2}r^2\times 0.8+2rh+0.8rh$	dM1	3.4
	$S=0.8r^2+2.8rh=0.8r^2+2.8\times\frac{600}{r}=0.8r^2+\frac{1680}{r}$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{dS}{dr}\right)=1.6r-\frac{1680}{r^2}$	M1 A1	3.1a 1.1b
	Sets $\frac{dS}{dr}=0\Rightarrow r^3=1050$ $r=$ awrt 10.2	dM1 A1	2.1 1.1b
		(4)	
(c)	Attempts to substitute their positive r into $\left(\frac{d^2S}{dr^2}\right)=1.6+\frac{3360}{r^3}$ and considers its value or sign	M1	1.1b
	E.g. Correct $\frac{d^2S}{dr^2}=1.6+\frac{3360}{r^3}$ with $\left.\frac{d^2S}{dr^2}\right\|_{r=10.2}=5>0$ proving a minimum value of S	A1	1.1b
		(2)	
			(10 marks)

Notes:

Volume $=0.4r^2h$



Total surface area $=2rh+0.8r^2+0.8rh$

(a)

M1: Attempts to use the fact that the volume of the toy is 240 cm^3

Sight of $\frac{1}{2}r^2 \times 0.8 \times h = 240$ leading to $h = ...$ or $rh = ...$ scores this mark

But condone an equation of the correct form so allow for $kr^2h = 240 \Rightarrow h = ...$ or $rh = ...$

A1: A correct expression for $h = \frac{600}{r^2}$ or $rh = \frac{600}{r}$ which may be left unsimplified.

This may be implied when you see an expression for S or part of S E.g $2rh = 2r \times \frac{600}{r^2}$

dM1: Attempts to substitute their $h = \frac{a}{r^2}$ o.e. such as $hr = \frac{a}{r}$ into a **correct** expression for S

Sight of $\frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + rh + rh + 0.8rh$ with an appropriate substitution

Simplified versions such as $0.8r^2 + 2rh + 0.8rh$ used with an appropriate substitution is fine.

A1*: Correct work leading to the given result.

$S =$, $SA =$ or surface area = must be seen at least once in the correct place

The method must be made clear so expect to see evidence. For example

$S = 0.8r^2 + 2rh + 0.8rh \Rightarrow S = 0.8r^2 + 2r \times \frac{600}{r^2} + 0.8r \times \frac{600}{r^2} \Rightarrow S = 0.8r^2 + \frac{1680}{r}$ would be fine.

(b) There is no requirement to see $\frac{dS}{dr}$ in part (b). It may even be called $\frac{dy}{dx}$.

M1: Achieves a derivative of the form $pr \pm \frac{q}{r^2}$ where p and q are non- zero constants

A1: Achieves $\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$

dM1: Sets or implies that their $\frac{dS}{dr} = 0$ and proceeds to $mr^3 = n, \quad m \times n > 0$. It is dependent upon a correct attempt at differentiation. This mark may be implied by a correct answer to their $pr - \frac{q}{r^2} = 0$

A1: $r =$ awrt 10.2 or $\sqrt[3]{1050}$

(c)

M1: Attempts to substitute their positive r (found in (b)) into $\left(\frac{d^2S}{dr^2} =\right) e \pm \frac{f}{r^3}$ where e and f are non zero and finds its value or sign.

Alternatively considers the sign of $\left(\frac{d^2S}{dr^2} =\right) e \pm \frac{f}{r^3}$ (at their positive r found in (b))

Condone the $\frac{d^2S}{dr^2}$ to be $\frac{d^2y}{dx^2}$ or being absent, but only for this mark.

A1: States that $\frac{d^2S}{dr^2}$ or $S'' = 1.6 + \frac{3360}{r^3} =$ awrt $5 > 0$ proving a minimum value of S

This is dependent upon having achieved $r =$ awrt 10 and a correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$

It can be argued without finding the value of $\frac{d^2S}{dr^2}$. E.g. $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3} > 0$ as $r > 0$, so minimum value of S. For consistency it is also dependent upon having achieved $r =$ awrt 10

Do **NOT** allow $\frac{d^2y}{dx^2}$ for this mark

12. The function f is defined by

$$f(x) = \frac{e^{3x}}{4x^2 + k}$$

where k is a positive constant.

(a) Show that

$$f'(x) = (12x^2 - 8x + 3k)\,g(x)$$

where $g(x)$ is a function to be found.

(3)

Given that the curve with equation $y = f(x)$ has at least one stationary point,

(b) find the range of possible values of k.

(3)

Question	Scheme	Marks	AOs
12(a)	$f(x)=\dfrac{e^{3x}}{4x^2+k} \Rightarrow f'(x)=\dfrac{\left(4x^2+k\right)3e^{3x}-8xe^{3x}}{\left(4x^2+k\right)^2}$ or $f(x)=e^{3x}\left(4x^2+k\right)^{-1} \Rightarrow f'(x)=3e^{3x}\left(4x^2+k\right)^{-1}-8xe^{3x}\left(4x^2+k\right)^{-2}$	M1 A1	1.1b 1.1b
	$f'(x)=\dfrac{\left(12x^2-8x+3k\right)e^{3x}}{\left(4x^2+k\right)^2}$	A1	2.1
		(3)	
(b)	If $y=f(x)$ has at least one stationary point then $12x^2-8x+3k=0$ has at least one root	B1	2.2a
	Applies $b^2-4ac\ (\geqslant)\ 0$ with $a=12, b=-8, c=3k$	M1	2.1
	$0<k\leqslant\dfrac{4}{9}$	A1	1.1b
		(3)	
			(6 marks)

Notes:

(a)

M1: Attempts the quotient rule to obtain an expression of the form $\dfrac{\alpha\left(4x^2+k\right)e^{3x}-\beta xe^{3x}}{\left(4x^2+k\right)^2}$, $\alpha\beta>0$

condoning bracketing errors/omissions as long as the intention is clear.
If the quotient rule formula is quoted it must be correct.

Condone e.g. $f'(x)=\dfrac{\left(4x^2+k\right)3e^{3x}-8xe^{3x}}{\left(4x^2+k\right)}$ provided an incorrect formula is not quoted.

May also see product rule applied to $e^{3x}\left(4x^2+k\right)^{-1}$ to obtain an expression of the form

$\alpha e^{3x}\left(4x^2+k\right)^{-1}+\beta xe^{3x}\left(4x^2+k\right)^{-2}$ $\alpha,\beta 0<0$ condoning bracketing errors/omissions as long as the intention is clear. If the product rule formula is quoted it must be correct.

A1: Correct differentiation in any form with correct bracketing which may be implied by subsequent work.

A1: Obtains $f'(x)=\left(12x^2-8x+3k\right)g(x)$ where $g(x)=\dfrac{e^{3x}}{\left(4x^2+k\right)^2}$ **or equivalent**

e.g. $g(x)=e^{3x}\left(4x^2+k\right)^{-2}$

Allow recovery from "invisible" brackets earlier and apply isw here once a correct answer is seen.
Note that the complete form of the answer is not given so allow candidates to go from e.g.

$\dfrac{\left(4x^2+k\right)3e^{3x}-8xe^{3x}}{\left(4x^2+k\right)^2}$ or $3e^{3x}\left(4x^2+k\right)^{-1}-8xe^{3x}\left(4x^2+k\right)^{-2}$ to $\dfrac{\left(12x^2-8x+3k\right)e^{3x}}{\left(4x^2+k\right)^2}$ for the final mark.

The "$f'(x) =$" must appear at some point but allow e.g. "$\frac{dy}{dx} =$"

(b) Note that B0M1A1 is not possible in (b)

B1: Deduces that if $y = f(x)$ has at least one stationary point then $12x^2 - 8x + 3k = 0$ has at least one root. There is no requirement to formally state $\frac{e^{3x}}{(4x^2+k)^2} > 0$

This may be implied by an attempt at $b^2 - 4ac \geqslant 0$ or $b^2 - 4ac > 0$ condoning slips.

M1: Attempts $b^2 - 4ac ... 0$ with $a = 12,\ b = -8,\ c = 3k$ where … is e.g. "=", <, >, etc.

Alternatively attempts to complete the square and sets rhs ...0

E.g. $12x^2 - 8x + 3k = 0 \Rightarrow x^2 - \frac{2}{3}x + \frac{1}{4}k = 0 \Rightarrow \left(x - \frac{1}{3}\right)^2 = \frac{1}{9} - \frac{1}{4}k$ leading to $\frac{1}{9} - \frac{1}{4}k \geqslant 0$

A1: $0 < k \leqslant \frac{4}{9}$ but condone $k \leqslant \frac{4}{9}$ and condone $0 \leqslant k \leqslant \frac{4}{9}$

Must be in terms of *k* not *x* so do not allow e.g. $0 < x \leqslant \frac{4}{9}$ but condone $\left(0,\ \frac{4}{9}\right]$ or $\left[0,\ \frac{4}{9}\right]$

16.

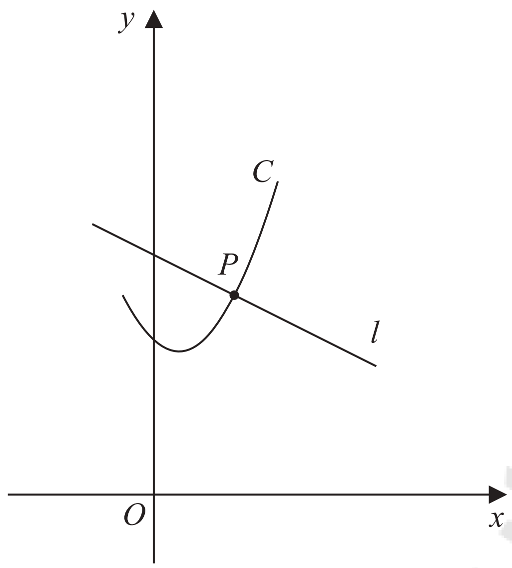


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

(2)

The straight line with equation

$$y = -\frac{1}{2}x + k \qquad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k.

(5)

Question	Scheme	Marks	AOs
16(a)	Attempts $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{4\sec^2 t\tan t}{2\sec^2 t}\left(=2\tan t\right)$	M1 A1	1.1b 1.1b
	At $t=\dfrac{\pi}{4}$, $\dfrac{dy}{dx}=2, x=3, y=7$	M1	2.1
	Attempts equation of normal $y-7=-\dfrac{1}{2}(x-3)$	M1	1.1b
	$y=-\dfrac{1}{2}x+\dfrac{17}{2}$ *	A1*	2.1
		(5)	
(b)	Attempts to use $\sec^2 t=1+\tan^2 t\Rightarrow\dfrac{y-3}{2}=1+\left(\dfrac{x-1}{2}\right)^2$	M1	3.1a
	$\Rightarrow y-3=2+\dfrac{(x-1)^2}{2}\Rightarrow y=\dfrac{1}{2}(x-1)^2+5$ *	A1*	2.1
		(2)	
	(b) Alternative 1:		
	$y=\dfrac{1}{2}(x-1)^2+5=\dfrac{1}{2}(2\tan t+1-1)^2+5$ $=\dfrac{1}{2}4\tan^2 t+5=2\left(\sec^2 t-1\right)+5$	M1	3.1a
	$=2\sec^2 t+3=y$*	A1	2.1
	(b) Alternative 2:		
	$x=2\tan t+1\Rightarrow t=\tan^{-1}\left(\dfrac{x-1}{2}\right)\Rightarrow y=2\sec^2\left(\tan^{-1}\left(\dfrac{x-1}{2}\right)\right)+3$ $\Rightarrow y=2\left(1+\tan^2\left(\tan^{-1}\left(\dfrac{x-1}{2}\right)\right)\right)+3$	M1	3.1a
	$\Rightarrow y=2\left(1+\left(\dfrac{x-1}{2}\right)^2\right)+3=\dfrac{1}{2}(x-1)^2+5$*	A1	2.1
	(b) Alternative 3:		
	$\dfrac{dy}{dx}=2\tan t=x-1\Rightarrow y=\int(x-1)dx=\dfrac{x^2}{2}-x+c$ $(3,7)\to 7=\dfrac{3^2}{2}-3+c\Rightarrow c=\dfrac{11}{2}$	M1	3.1a
	$\dfrac{x^2}{2}-x+\dfrac{11}{2}=\dfrac{1}{2}\left(x^2-2x\right)+\dfrac{11}{2}=\dfrac{1}{2}(x-1)^2-\dfrac{1}{2}+\dfrac{11}{2}=\dfrac{1}{2}(x-1)^2+5$*	A1	2.1

(c)	**Attempts the lower limit for k:** $\frac{1}{2}(x-1)^2+5=-\frac{1}{2}x+k \Rightarrow x^2-x+(11-2k)=0$ $b^2-4ac=1-4(11-2k)=0 \Rightarrow k=\ldots$	M1	2.1
	$(k=)\frac{43}{8}$	A1	1.1b
	Attempts the upper limit for k: $(x,y)_{t=-\frac{\pi}{4}}: t=-\frac{\pi}{4} \Rightarrow x=2\tan\left(-\frac{\pi}{4}\right)+1=-1,\ y=2\sec^2\left(-\frac{\pi}{4}\right)+3=7$ $(-1,\ 7),\ y=-\frac{1}{2}x+k \Rightarrow 7=\frac{1}{2}+k \Rightarrow k=\ldots$	M1	2.1
	$(k=)\frac{13}{2}$	A1	1.1b
	$\frac{43}{8}<k\leqslant\frac{13}{2}$	A1	2.2a
		(5)	
(12 marks)			
Notes:			

(a) **Must use parametric differentiation to score any marks in this part and not e.g. Cartesian form**

M1: For the key step of attempting $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$. There must be some attempt to differentiate both parameters however poor and divide the right way round so using $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}$ scores M0.

This may be implied by e.g. $\frac{\mathrm{d}x}{\mathrm{d}t}=2\sec^2 t,\ \frac{\mathrm{d}y}{\mathrm{d}t}=4\sec^2 t\tan t,\ t=\frac{\pi}{4} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=4,\ \frac{\mathrm{d}y}{\mathrm{d}t}=8 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=2$

A1: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4\sec^2 t\tan t}{2\sec^2 t}$. Correct expression in any form. May be implied as above.

Condone the confusion with variables as long as the intention is clear e.g.

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{4\sec^2 x\tan x}{2\sec^2 x}(=2\tan x)$ and allow subsequent marks if this is interpreted correctly

M1: For attempting to find the values of x, y and the gradient at $t=\frac{\pi}{4}$ **AND** getting at least two correct.

Follow through on their $\frac{\mathrm{d}y}{\mathrm{d}x}$ so allow for any two of $x=3,\ y=7,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2\left(\text{or their } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ at } t=\frac{\pi}{4}\right)$

Note that the $x=3,\ y=7$ may be seen as e.g. (3, 7) on the diagram. There must be a non-trivial $\frac{\mathrm{d}y}{\mathrm{d}x}$ for this mark e.g. they must have a $\frac{\mathrm{d}y}{\mathrm{d}x}$ to substitute into.

M1: For a correct attempt at the normal equation using their x and y at $t=\frac{\pi}{4}$ with the negative reciprocal of their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t=\frac{\pi}{4}$ having made some attempt at $\frac{\mathrm{d}y}{\mathrm{d}x}$ and all correctly placed.

For attempts using $y=mx+c$ they must reach as far as a value for c using their x and y at $t=\frac{\pi}{4}$ with the negative reciprocal of their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t=\frac{\pi}{4}$ all correctly placed.

A1*: Proceeds with a clear argument to the given answer with no errors.

(b)

M1: Attempts to use $\sec^2 t = 1 + \tan^2 t$ oe to obtain an equation involving y and $(x-1)^2$

E.g. as above or e.g. $y = 2\sec^2 t + 3 = 2\left(1+\tan^2 t\right) + 3 = 2\left(1+\left(\frac{x-1}{2}\right)^2\right) + 3$ for M1 and then

$y = \frac{1}{2}(x-1)^2 + 5*$ for A1

A1*: Proceeds with a clear argument to the given answer with no errors

Alternative 1:

M1: Uses the given result, substitutes for x and attempts to use $\sec^2 t = 1+\tan^2 t$ oe

A1: Proceeds with a clear argument to the y parameter and makes a (minimal) conclusion e.g. " $= y$" QED, hence proven etc.

Alternative 2:

M1: Uses the x parameter to obtain t in terms of arctan, substitutes into y and attempts to use $\sec^2 t = 1+\tan^2 t$ oe

A1: Proceeds with a clear argument to the given answer with no errors

Alternative 3:

M1: Uses $\frac{dy}{dx}$ from part (a) to express $\frac{dy}{dx}$ in terms of x, integrates and uses (3, 7) to find "c" to reach a Cartesian equation.

A1: Proceeds with a clear argument to the given answer with no errors

Allow the marks for (b) to score anywhere in their solution e.g. if they find the Cartesian equation in part (a)

(c)

M1: A full attempt to find the **lower** limit for k.

$\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11-2k) = 0 \Rightarrow b^2 - 4ac = 1 - 4(11-2k) = 0 \Rightarrow k = ...$

Score **M1** for setting $\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k$, rearranging to 3TQ form and attempts $b^2 - 4ac ... 0$ e.g. $b^2 - 4ac > 0$ or e.g. $b^2 - 4ac < 0$ correctly to find a value for k.

A1: $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

An alternative method using calculus for lower limit:

$$y = \frac{1}{2}(x-1)^2 + 5 \Rightarrow \frac{dy}{dx} = x - 1,\ x - 1 = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

$$x = \frac{1}{2} \Rightarrow y = \frac{1}{2}\left(\frac{1}{2} - 1\right)^2 + 5 = \frac{41}{8}$$

$$y = -\frac{1}{2}x + k \Rightarrow \frac{41}{8} = -\frac{1}{4} + k \Rightarrow k = ...$$

Score **M1** for $\frac{dy}{dx} =$ "a linear expression in x", sets $= -\frac{1}{2}$, solves a linear equation to find x and then substitutes into the given result in (b) to find y and then uses $y = -\frac{1}{2}x + k$ to find a value for k. **A1**: $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

An alternative method using parameters for lower limit:

$$y=-\frac{1}{2}x+k \Rightarrow 2\sec^2 t+3=-\frac{1}{2}(2\tan t+1)+k$$

$$\Rightarrow 2\left(1+\tan^2 t\right)+3=-\frac{1}{2}(2\tan t+1)+k \Rightarrow 2\tan^2 t+\tan t+5.5-k=0$$

$$b^2-4ac=0 \Rightarrow 1-4\times 2(5.5-k)=0 \Rightarrow k=\frac{43}{8}$$

Score **M1** for substituting parametric form of x and y into $y=-\frac{1}{2}x+k$, uses $\sec^2 t=1+\tan^2 t$ rearranges to 3TQ form and attempts $b^2-4ac...0$ or e.g. $b^2-4ac>0$ or $b^2-4ac<0$ correctly to find a value for k.

A1: $k=\frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k>\frac{43}{8}$, $k<\frac{43}{8}$

M1: A full attempt to find the **upper** limit for k. This requires an attempt to find the value of x and the value of y using $t=-\frac{\pi}{4}$, the substitution of these values into $y=-\frac{1}{2}x+k$ and solves for k.

A1: $k=\frac{13}{2}$. Look for this value e.g. may appear in an inequality.

A1: Deduces the correct range for k: $\frac{43}{8}<k\leqslant\frac{13}{2}$

Allow equivalent notation e.g. $\left(k\leqslant\frac{13}{2} \text{ and } k>\frac{43}{8}\right)$, $\left(k\leqslant\frac{13}{2}\cap k>\frac{43}{8}\right)$, $\left(\frac{43}{8},\frac{13}{2}\right]$

But not e.g. $\left(k\leqslant\frac{13}{2},\ k>\frac{43}{8}\right)$, $\left(k\leqslant\frac{13}{2}\cup k>\frac{43}{8}\right)$, $\left(k\leqslant\frac{13}{2} \text{ or } k>\frac{43}{8}\right)$ and do not allow if in terms of x.

Allow equivalent exact values for $\frac{43}{8}$, $\frac{13}{2}$

There may be other methods for finding the upper limit which are valid. If you are in any doubt if a method deserves credit then use Review.